# Self-assembling electrical connections based on the principle of minimum resistance

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We study self-constructing and self-repairing electrical connections built by agglomeration of metallic particles between two electrodes. Our experiments show that self-assembling electrical connections grow by building a chain of particles between two electrodes immersed in a dielectric liquid. We find that the growth time for the self-assembling process is a linear function of the initial average spacing of metallic particles and a linear function of the distance between the electrodes. Furthermore, the experiments demonstrate the ability of the electrical connection to self-repair following small perturbations. We show that the agglomeration process occurs in such a way as to minimize the overall resistance of the system. We discuss possible future applications of this phenomenon for fabricating nanoscale circuits. [S1063-651X(96)06407-0]

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## I. INTRODUCTION

The neural system of the brain is adaptive, self-repairing, and self-assembling [1]. These properties are some of the most distinctive differences between the information processing structures of living beings and those of current computer hardware. Over the past two decades a great deal of scientific effort has been devoted to further understand the problems of pattern formation, dendritic solidification, and to discover unifying paradigms for the dynamics and attractors of open nonlinear systems. The principles of extremal entropy production [2], self-organized criticality [3–6], marginal stability [7], and minimum resistance [8,9] have been used to help explain these phenomena. Until now, these general paradigms have rarely been implemented to design materials and devices which could (1) efficiently respond to large destructive perturbations, (2) effectively adapt to a changing environment, or (3) solve complex new tasks such as the construction of neural networks. However, there is an urgent need for such adaptive and creative materials and devices. Rapid progress in nanotechnology makes it possible to manufacture electronic devices on a quantum scale, e.g., single electron tunneling transistors [10]. If such nanodevices are densely packed in two or possibly three dimensions it seems practically impossible to design adequate noise-resistant circuitry or to test the functionality of every element within these complex devices. Therefore it would be highly desirable for computer hardware to independently perform the wiring according to a given task and to self-repair when errors occur.

In our previous work we investigated the agglomeration process of metallic particles under the influence of an electric current. We demonstrated that it is possible to obtain self-organized dendritic structures using a two-dimensional electrode and a point electrode [11]. These structures are stable, possess a fractal geometry, and satisfy the principle of minimum resistance.

In this paper we investigate the agglomeration process of metallic particles which are suspended between two point electrodes in a highly viscous oil. We find that under certain conditions the agglomeration process results in a conducting chain of metallic particles, which in turn may be considered a self-assembled wire. We further examine the growth time of the emerging structures as a function of the concentration of particles in the oil and of the distance between the electrodes. Section II of this report gives a description of the experimental setup, followed by an overview of the experimental results in Sec. III. In Sec. IV the principle of minimum resistance is discussed. Finally, Sec. V provides a brief analysis of the results and possible future applications for self-assembling electrical connections.

## II. THE EXPERIMENT

In the setup of the experiment, $N$ smooth steel spheres (radius $r=1$ mm, mass $m=33.0\pm1.3$ mg) are distributed at the bottom of a cylindrical cell such that the total number of spheres within a given area of $A=1$ cm$^2$ is constant. We refer to the average number of spheres per cm$^2$ as the concentration of particles $C_p$. The cylindrical cell (radius $r_c=70$ mm) consists of a thin layer of oil (height $h=5$ mm) within an acrylic dish. Two needle-shaped electrodes $a$ and $b$ (maximum radius $r_e=0.5$ mm) are placed in the cell with a fixed distance $d$ separating their tips. Electrode $a$ is grounded throughout the experiment while a potential $V$ is initiated through electrode $b$. This potential is adjustable between $V=10$ kV and $V=25$ kV. A schematic of the setup is shown in Fig. 1.

The working fluid which makes up the cell is castor oil, chosen for its small conductivity [$\sigma_{\rm oil}(20\,°{\rm C})\leqslant10^{-12}$ $(\Omega{\rm m})^{-1}$], high dielectric constant ($\kappa_{\rm oil}\approx4.7$), and high viscosity [$\eta(20\,°{\rm C})=0.990\pm1\%$ Pa s]. The high dielectric constant and relatively small conductivity allow a large amount of electrostatic energy to be stored in the oil, while heating due to Ohmic resistance is kept to a minimum. The high viscosity of the oil causes the motion of the particles to be slow in comparison with the relaxation time of the distribution of the electric charges. This property allows for a separation of time scales which is useful in modeling the dynamics of the experiment.

For times $t<0$ the spheres lie motionless in the cylindrical cell. They are distributed in one of two ways: (1) ran-

 

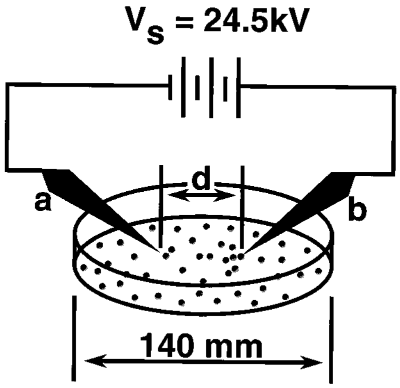


FIG. 1. Experimental setup.

domly with an average separation $\langle r_p \rangle$ between neighboring spheres or (2) evenly on a square grid. At time $t=0$ a potential of approximately $V=24$ kV is supplied to the electrodes. The applied voltage induces an electric field that causes the metallic spheres within the vicinity of the electrodes to agglomerate in order to reduce the resistance of the system. Initially, the resistance between the electrodes is approximately the resistance of the castor oil. It drops by a factor of approximately $10^{11}$ almost instantaneously when a chain of particles connects the two electrodes. We measure the growth time $T_c$ from the beginning of the experiment to the sudden drop in resistance. Typical growth processes are represented by series of pictures in Figs. 2 and 3.

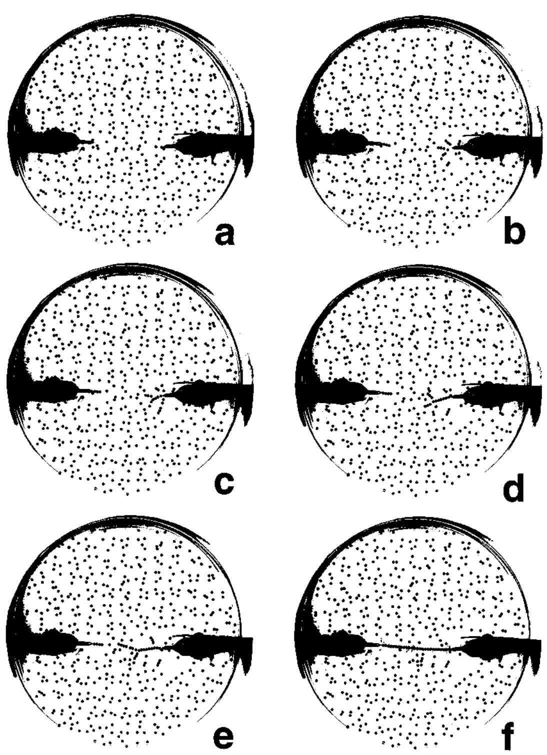


FIG. 2. Example of an agglomeration process at different time stages for randomly distributed particles. (a) $t=0$ s, (b) $t=20$ s, (c) $t=30$ s, (d) $t=40$ s, (e) $t=60$ s, (f) $t=77$ s.

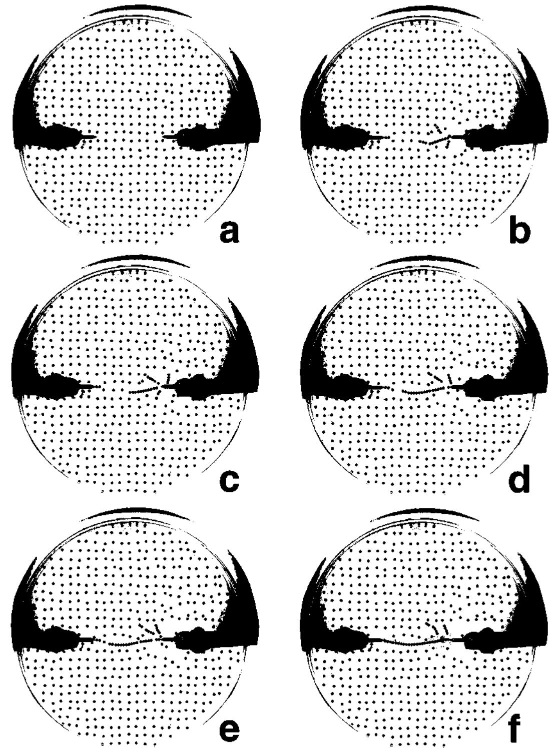


FIG. 3. Example of an agglomeration process at different time stages for particles arranged on a grid. (a) $t=0$ s, (b) $t=30$ s, (c) $t=45$ s, (d) $t=65$ s, (e) $t=75$ s, (f) $t=85$ s.

### III. DYNAMICS OF THE AGGLOMERATION PROCESS

When the voltage $V$ is initially applied to the electrodes, the metallic spheres in the vicinity of the electrodes are momentarily repelled. Approximately half a second later the particles are attracted by the electrodes where they begin to line up in almost straight chains. Typically, two to four small chains are observed to grow from a given electrode. As the growth process continues, branches emanating from an electrode strongly compete against each other. The dominating branch of one electrode moves toward the respective branch from the other electrode. This agglomerated row of metallic spheres tends to align itself along the shortest distance between the two electrodes. At this stage, if any gaps are present in the alignment of individual spheres, additional spheres in the vicinity are also attracted to the chain of particles to complete a circuit of minimal resistance.

The growth time $T_c$ from the beginning of each experiment to the moment of the formation of a completed chain is measured. The arithmetic average $\overline{T_c}$ of the growth time $T_c$ and the standard deviation $\sigma_c$ are evaluated for ten measurements. In a cell consisting of randomly distributed particles at an average concentration $C_p$, the average $\overline{T_c}$ is found to be directly proportional to the distance $d$ between the two electrodes (see Fig. 4). An interesting observation is the fact that the time $T_c$ is affected by the symmetry of the growing chains. When the branches from an electrode align themselves in a symmetrical manner along the axis defined by the end points of the two electrodes, the growth time $T_c$ is approximately 5 to 10 longer than $\overline{T_c}+\sigma_c$. When irregular

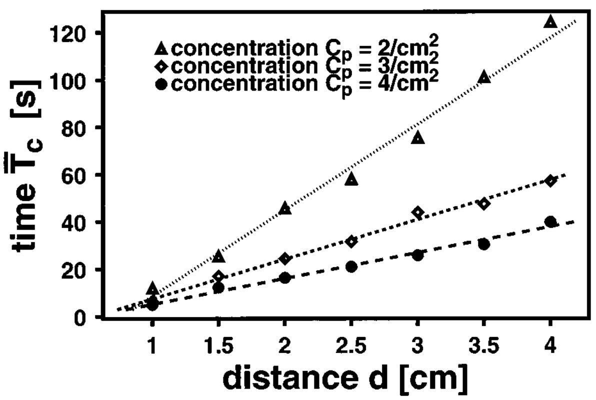


FIG. 4. Average time $\overline{T_c}$ as a function of the distance $d$ between the electrodes for varying particle concentrations $C_p$.

patterns of chains develop the growth time $T_c$ is found to be in the range of $\overline{T_c} \pm \sigma_c$.

The experiments were repeated with the metallic spheres arranged on a rectangular grid (see Fig. 3). In this setup, the agglomeration occurs more slowly than in experiments with randomly arranged particles (see Fig. 5). This effect is not evident for distances smaller than $d=2$ cm.

Additional experiments were performed with a constant distance $d$ between the electrodes, while the particle concentrations $C_p$ were varied. The data show that $\overline{T_c}$ are proportional to the expectation value for the average spacing of randomly distributed spheres, $\langle r_p \rangle$. The results are presented in Fig. 6.

The behavior of the self-assembled electrical connections following small perturbations was also investigated. First an electrical connection was established. The voltage was then switched off and the chain of spheres was slightly disturbed. After the voltage was reapplied to the electrodes, the removed particles returned to their original position in less than 2 s (see Fig. 7). The behavior of the metallic spheres suggests that their dynamics may be described in terms of a variational principle. In the following section, we show that the dynamics of self-assembling electrical connections minimizes the resistance of the system.

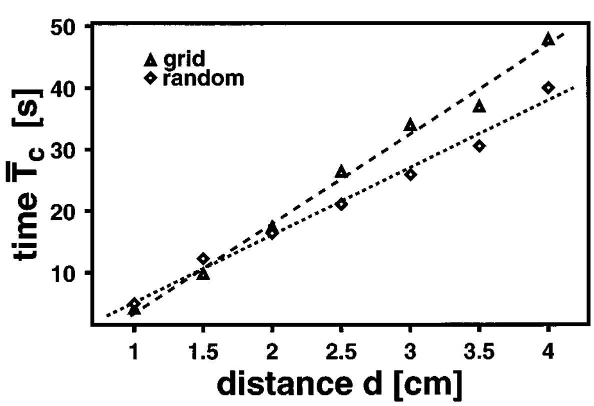


FIG. 5. Average time $\overline{T_c}$ as a function of the distance $d$ between the electrodes for a particle concentration of $C_p = 4/\mathrm{cm}^2$. Short dashes represent experiments with randomly arranged particles and long dashes represent experiments with particles arranged on a rectangular grid.

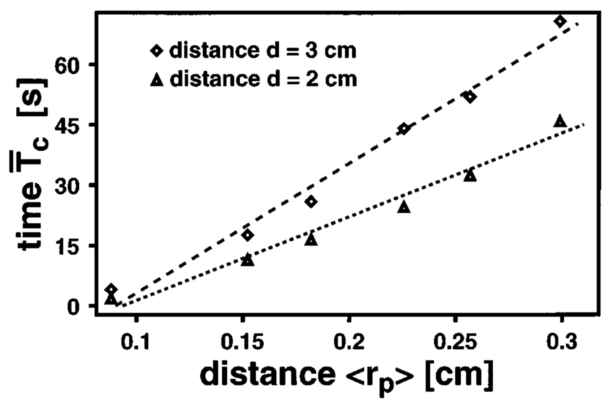


FIG. 6. Average time $\overline{T_c}$ as a function of the average spacing $\langle r_p \rangle$ of particles for distances $d=2$ cm (short dashed line) and $d=3$ cm (long dashed line) between the electrodes.

## IV. THE PRINCIPLE OF MINIMUM RESISTANCE

To model the dynamics of the experiment, we consider the two-dimensional motion of metallic spheres in the horizontal plane of the dish. The experimental system has spatially fixed boundary conditions consisting of two point electrodes located at $x_1, y_1$ and $x_2, y_2$, respectively, between which a potential difference $V = \Phi(x_1, y_1) - \Phi(x_2, y_2)$ is induced. The perimeter of the cylindrical cell is insulated, and $\vec{\nabla}\Phi(r_{\mathrm{boundary}}) = 0$. The total resistance $R_{\mathrm{tot}}$, is defined as the ratio of the potential difference $V$ to the total current $I_{\mathrm{tot}}$ flowing from one electrode to another. In the following, we assume that all of the current flowing out of one electrode enters the other electrode, i.e., $I_1 = -I_2 = I_{\mathrm{tot}}$. After defining a few relations, we examine the temporal behavior of the total resistance $R_{\mathrm{tot}}$ and demonstrate that $dR_{\mathrm{tot}}/dt \leqslant 0$.

The continuity equation

$$\vec{\nabla} \cdot \vec{J}(x,y) + \frac{\partial \rho(x,y)}{\partial t} = 0 \tag{1}$$

expresses the local conservation of charge in the system. In the experiment, the motion of the metallic spheres is heavily damped by the viscous castor oil. The motion of the spheres is slow compared to the rate of charge relaxation $\sigma_{\mathrm{oil}}/\epsilon_{\mathrm{oil}}$ of the oil. Therefore the time dependence of the charge density $\rho(x,y)$ may be neglected through adiabatic elimination [12]. We then have the steady current condition

$$\vec{\nabla} \cdot \vec{J}(x,y) = 0. \tag{2}$$

Since the spheres are heavily damped, we neglect inertial forces and introduce an equation of motion of the form

$$\vec{F}_i = \gamma \dot{\vec{r}}_i = \frac{1}{2\epsilon_{\mathrm{oil}}} \int_S \vec{E}_i^2 da_i \hat{n}_i \quad (i = 1, \ldots, N), \tag{3}$$

where $\gamma$ is an effective friction coefficient, $\dot{\vec{r}}$ is the velocity of sphere $i$, $\epsilon_{\mathrm{oil}}$ is the permittivity of the oil, $\vec{E}(x,y) = -\vec{\nabla}\Phi(x,y)$ is the electric field at the surface of the

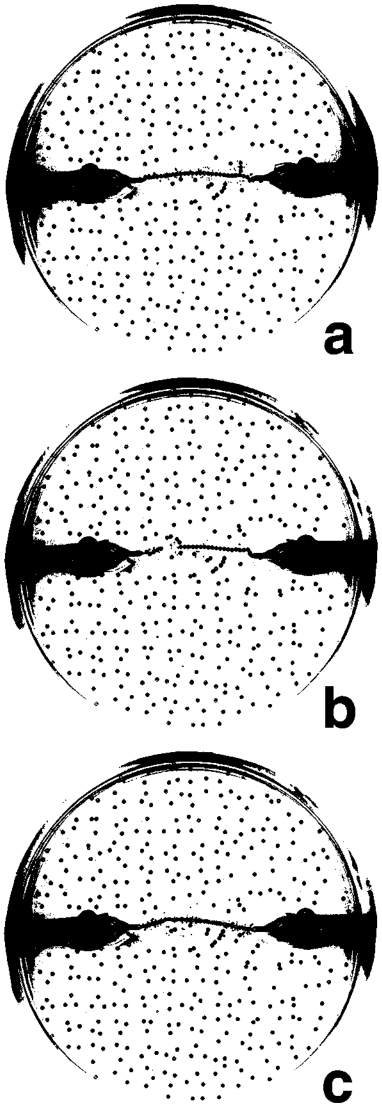

FIG. 7. Example of a self-repairing electrical connection. (a) connection before the destructive perturbation, (b) connection at time of perturbation, (c) connection 2 s after the perturbation.

conducting sphere [14], and the integral is taken over the surface of the sphere, where $\hat{n}_i$ is its normal.

We consider Ohmic media, for which the current density is

$$\vec{J}(x,y) = \sigma \vec{E}(x,y), \tag{4}$$

where $\sigma$ is the local isotropic conductivity. As shown in Appendix B, the total charge on an isolated sphere (one which is not connected to the battery terminals) is zero.

If there exists a potential difference $\Phi_{ij} = \Phi(x_i, y_i) - \Phi(x_j, y_j)$ between spheres $i$ and $j$, a current will flow between them according to

$$\Phi_{ij} = I_{ij} R_{ij}, \tag{5}$$

where $I_{ij}$ is the current which leaves sphere $i$ and enters sphere $j$ directly. $R_{ij}$ is the resistance of the current $I_{ij}$ between spheres $i$ and $j$, and $R_{ij} = R_{ji}$.

We introduce a two-dimensional model to describe the dynamics of the conductors. For two circular conductors in two dimensions, the electric field distribution can be found exactly by using the method of images. From this result, we show in Appendix C that the force on each of the two conductors may be written in terms of the gradient of the total resistance between conductors with respect to the location of each conductor:

$$\vec{F}_{ij} = -\frac{\epsilon}{2\sigma} I_{ij}^2 \vec{\nabla}_{r_i} R_{ij} \quad (i,j=1,2; i \neq j). \tag{6}$$

Furthermore, in Appendix D we show that the force between a point charge and a sphere may be written using an expression similar to the one above. This result is used for the force-resistance relation for a single conducting sphere and a point electrode.

To find the total force on a given sphere $i$, we then make the following approximation: we add vectorially the two-body forces between sphere $i$ and the other conducting spheres, each of which is written in terms of the gradient of the resistance between spheres. In doing so, the resistance between spheres is defined in terms of the total current $J_{ij} = \sigma E_{ij}$ which flows directly from sphere $i$ to sphere $j$.

The total force on a given conducting sphere $i$ is written

$$\vec{F}_i = -\sum_{j, j\neq i} \frac{\epsilon}{2\sigma} I_{ij}^2 \vec{\nabla}_{r_i} R_{ij}. \tag{7}$$

With Eq. 3, the equation of motion becomes

$$\gamma \dot{\vec{r}}_i = -\sum_{j, j\neq i} \frac{\epsilon}{2\sigma} I_{ij}^2 \vec{\nabla}_{r_i} R_{ij}, \tag{8}$$

where $\gamma$ is the effective viscosity. Since the two electrodes are held fixed, we also have $\dot{r}_1 = 0$ and $\dot{r}_2 = 0$.

Using the relation, derived in Appendix A of this manuscript,

$$\frac{I_{ij}^2}{I_{\rm tot}^2} = \frac{\partial R_{\rm tot}}{\partial R_{ij}} \tag{9}$$

the equation of motion becomes

$$\dot{\vec{r}}_i = -\frac{\epsilon}{2\gamma\sigma} I_{\rm tot}^2 \sum_{j, j\neq i} \frac{\partial R_{\rm tot}}{\partial R_{ij}} \left( \frac{\partial R_{ij}}{\partial x_i} \hat{x} + \frac{\partial R_{ij}}{\partial y_i} \hat{y} \right). \tag{10}$$

We now consider the time dependence of the total resistance between the fixed electrodes. Due to the separation of time scales, the total resistance $R_{\rm tot}$ is not an explicit function of time, but depends only on the locations $\vec{r}_i$ of the conductors. The total time derivative of $R_{\rm tot}$ is written in terms of partial derivatives with respect to the sphere locations $\vec{r}_i = x_i \hat{x} + y_i \hat{y}$.

$$\frac{dR_{\rm tot}}{dt} = \frac{1}{2} \sum_i \sum_{j, j\neq i} \frac{\partial R_{\rm tot}}{\partial R_{ij}} \left( \frac{\partial R_{ij}}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial R_{ij}}{\partial y_i} \frac{dy_i}{dt} + \frac{\partial R_{ij}}{\partial x_j} \frac{dx_j}{dt} + \frac{\partial R_{ij}}{\partial y_j} \frac{dy_j}{dt} \right). \tag{11}$$

Substituting Eq. (10) for the velocities of conductors $i,j$, we have

$$\frac{dR_{\mathrm{tot}}}{dt}=-\frac{\epsilon I_{\mathrm{tot}}^2}{4\gamma\sigma}\sum_i\sum_{j,j\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\left[\frac{\partial R_{ij}}{\partial x_i}\left(\sum_{k,k\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ik}}\frac{\partial R_{ik}}{\partial x_i}\right)+\frac{\partial R_{ij}}{\partial y_i}\left(\sum_{k,k\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ik}}\frac{\partial R_{ik}}{\partial y_i}\right)+\frac{\partial R_{ij}}{\partial x_j}\left(\sum_{m,m\neq j}\frac{\partial R_{\mathrm{tot}}}{\partial R_{jm}}\frac{\partial R_{jm}}{\partial x_j}\right)+\frac{\partial R_{ij}}{\partial y_j}\left(\sum_{m,m\neq j}\frac{\partial R_{\mathrm{tot}}}{\partial R_{jm}}\frac{\partial R_{jm}}{\partial y_j}\right)\right]. \tag{12}$$

Rewriting Eq. (12) we have

$$\frac{dR_{\mathrm{tot}}}{dt}=-\frac{\epsilon I_{\mathrm{tot}}^2}{4\gamma\sigma}\left[\sum_i\left(\sum_{j,j\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial x_i}\right)\left(\sum_{k,k\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ik}}\frac{\partial R_{ik}}{\partial x_i}\right)+\sum_i\left(\sum_{j,j\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial y_i}\right)\left(\sum_{k,k\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ik}}\frac{\partial R_{ik}}{\partial y_i}\right)+\sum_{j,j\neq i}\left(\sum_i\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial x_j}\right)\times\left(\sum_{m,m\neq j}\frac{\partial R_{\mathrm{tot}}}{\partial R_{jm}}\frac{\partial R_{jm}}{\partial x_j}\right)+\sum_{j,j\neq i}\left(\sum_i\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial y_j}\right)\left(\sum_{m,m\neq j}\frac{\partial R_{\mathrm{tot}}}{\partial R_{jm}}\frac{\partial R_{jm}}{\partial y_j}\right)\right]. \tag{13}$$

Since $j$ and $k$ are dummy indices in the first and second products of Eq. 13, and $i$ and $m$ are dummy indices in the third and fourth products, and using the fact that $R_{ij}=R_{ji}$, we find the time derivative of the total resistance is

$$\frac{dR_{\mathrm{tot}}}{dt}=-\frac{\epsilon I_{\mathrm{tot}}^2}{4\gamma\sigma}\left\{\sum_i\left[\left(\sum_{j,j\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial x_i}\right)^2+\left(\sum_{j,j\neq i}\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial y_i}\right)^2\right]+\sum_{j,j\neq i}\left[\left(\sum_i\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial x_j}\right)^2+\left(\sum_i\frac{\partial R_{\mathrm{tot}}}{\partial R_{ij}}\frac{\partial R_{ij}}{\partial y_j}\right)^2\right]\right\}. \tag{14}$$

Thus we see that $\dot{R}_{\mathrm{tot}}\leq 0$; that is, $R_{\mathrm{tot}}$ is a Lyapunov function for the dynamics of the system.

## V. DISCUSSION

We have presented the experimental and theoretical investigation of self-assembling electrical connections. We found linear relationships between the distance $d$ separating the electrodes and the average growth time of the connections $\overline{T_c}$. The experimental data suggest that the average initial separation of particles $\langle r_p\rangle$ is directly proportional to $\overline{T_c}$. In addition we found that self-assembled wires are stable, reconnecting quickly when the chains are broken. We further observed that the agglomeration is comparably slower when symmetrical patterns of chains develop during the growth process. We propose that these patterns are states of unstable equilibrium.

We also found that the speed of the growth process depends on the initial arrangement of the particles. This observation may be due to the fact that the expectation value for the distance to the next neighbor for randomly distributed particles, $\langle r_p\rangle$, is smaller than the distance between spheres on a rectangular grid. For small distances $d$ between the electrodes the characteristic longer time $\overline{T_c}$ is not evident for uniformly distributed particles. This may be due to the fact that particles located in the vicinity of the electrodes are momentarily repelled at the beginning of the experiment. It can be assumed that at relatively small distances $d$ this causes uniformly placed particles to be randomly distributed in the area affected throughout the experiment. In turn, this would yield growth times similar to those experiments starting with randomly distributed spheres.

In addition, we have shown, using a simple two-dimensional model for the dynamics of the particles, that the total resistance between electrodes is a Lyapunov function. For some simple open systems, we derived an expression for the force on a conductor in terms of a gradient of the total resistance and current in the system:

$$\vec{F}=-\frac{\epsilon}{2\sigma}I_{\mathrm{tot}}^2\vec{\nabla}R_{\mathrm{tot}}. \tag{15}$$

We suggest that this relation may be useful to describe the dynamics of an open, linearly dissipative system when inertial forces may be neglected.

As an example of the applicability of this relation, we consider the problem of a leaky parallel-plate capacitor filled with an Ohmic dielectric. The force between capacitor plates may be written

$$\vec{F}_i=\tfrac{1}{2}\epsilon|\vec{E}|^2A_i\hat{n}_i \quad (i=1,2), \tag{16}$$

where $\epsilon$ is the permittivity of the dielectric, $\vec{E}=-\vec{\nabla}\Phi$ is the electric field at the surface of plate $i$, $A_i$ is the area of plate $i$, and $\hat{n}_i$ is its outward normal. We consider general position-dependent boundary conditions, such that the potential difference between the plates, $\Phi_2-\Phi_1=\Delta\Phi$, is an arbitrary, continuous function of the plate separation, $x_2-x_1$, i.e., $\Delta\Phi=\Delta\Phi(x_2-x_1)$. The electric field between the plates is

$$\vec{E}(x_2-x_1)=-\frac{\Delta\Phi(x_2-x_1)}{x_2-x_1}\hat{x} \tag{17}$$

and the force between them is

$$\vec{F}_i(x_2-x_1)=\frac{1}{2}\epsilon\left(-\frac{\Delta\Phi(x_2-x_1)}{x_2-x_1}\right)^2A_i\hat{n}_i. \tag{18}$$

The force may also be described in terms of the resistance

$$R_{\mathrm{tot}}(x_2-x_1)=\frac{\Delta\Phi(x_2-x_1)}{I_{\mathrm{tot}}}=\frac{\Delta\Phi(x_2-x_1)}{\sigma|\vec{E}|A_i}=\frac{x_2-x_1}{\sigma A}, \tag{19}$$

with $A_2=A_1=A$. Using our relation, Eq. (15), and $I_{\mathrm{tot}}=\sigma|\vec{E}|A$, we find the force between the plates as

$$\vec{F}_i = -\frac{\epsilon}{2\sigma} I_{\rm tot}^2 \vec{\nabla} R_{\rm tot} = \frac{1}{2}\epsilon |\vec{E}|^2 A_i \hat{n}_i \quad (i=1,2) \tag{20}$$

in agreement with Eq. (16). The dynamics of the capacitor plates may be determined through the minimization of the resistance, regardless of the nature (e.g., constant potential or constant charge) of the boundary conditions. This result is in stark contrast to the theory of minimal entropy production [2], for in our leaky capacitor example, the dissipation $P = I_{\rm tot}^2 R_{\rm tot}$ decreases as the plates move together when the charges are held constant, but increases if the potential difference is held constant.

In this paper we have experimentally and theoretically described the dynamics of self-assembling macroscopic wires. We propose to further investigate the dynamics of polar atoms in periodic potential wells. In particular, agglomerates of gold atoms between electrodes on silicon surfaces are expected to form stable nanowires at much smaller voltages. This phenomenon could be utilized to build self-assembling circuitry in densely packed complex electronic devices. These circuits are expected to be more tolerant of errors than contemporary nanocircuits. They also could be easily modified by regrowing the electrical connections according to a given task.

## ACKNOWLEDGMENTS

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## APPENDIX A: DERIVATION OF THE CURRENT-RESISTANCE RELATION

$$\frac{I_{kl}^2}{I_{\rm tot}^2} = \frac{\partial R_{\rm tot}}{\partial R_{kl}}. \tag{A1}$$

We consider a resistive network of $N$ nodes and $b$ branches, in which pairs of nodes are connected by single linear resistors $R_{ij}$, and $i,j=(1,\ldots,N)$. An arbitrary network of linear resistors may be represented this way, through series and parallel additions of resistors between nodes. We consider a single source of emf, $V_s$, within a branch between two arbitrary nodes of the network. The total power dissipated in the network is the sum of the Ohmic dissipation in each branch,

$$P_{\rm tot} = \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} I_{ij}^2 R_{ij}. \tag{A2}$$

The total resistance of the network between the two selected nodes bordering $V_s$ is $R_{\rm tot}$, and the total potential difference between the two nodes at any instant is $V_s = I_{\rm tot} R_{\rm tot}$. Thus the total Ohmic dissipation is

$$V_s I_{\rm tot} = I_{\rm tot}^2 R_{\rm tot} = \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} I_{ij}^2 R_{ij}. \tag{A3}$$

The above equation is known as Tellegen's theorem [15], which expresses the conservation of power. Taking the partial derivative of the above equation with respect to a given resistance $R_{kl}$, we obtain

$$2\frac{\partial I_{\rm tot}}{\partial R_{kl}} I_{\rm tot} R_{\rm tot} + I_{\rm tot}^2 \frac{\partial R_{\rm tot}}{\partial R_{kl}} = \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} 2\frac{\partial I_{ij}}{\partial R_{kl}} I_{ij} R_{ij} + I_{kl}^2. \tag{A4}$$

The procedure for the proof of the equivalence of the second term

$$T_2 = I_{\rm tot}^2 \frac{\partial R_{\rm tot}}{\partial R_{kl}} \tag{A5}$$

and fourth term

$$T_4 = I_{kl}^2 \tag{A6}$$

consists of the following steps.

(1) Label the nodes of the network $i=1,\ldots,N$ arbitrarily. Define the current which leaves node $i$ and enters node $j$ as the branch current $I_{ij}$. For the purposes of this proof, and for the solution of the network, the chosen directions of the branch currents need not be the directions of the currents in the actual physical system.

(2) Choose a tree $\mathcal{T}$ [15] of the graph $\mathcal{G}$ of the network, such that $\mathcal{T}$ does not include the branch which contains the source of emf, $V_s$. Determine the fundamental loops corresponding to the links of the tree $\mathcal{T}$. We denote each loop $\alpha$ as a set of indices $\{\alpha_q\}(q=1,\ldots,n)$, where each $\alpha_q$ represents a node of the network and appears only once in the set $\{\alpha_q\}$, except for the first, $\alpha_1$, which appears twice:

$$\alpha = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_{n+1} = \alpha_1\}. \tag{A7}$$

For each fundamental loop $\alpha$, define a loop current $I_\alpha$ with the associated reference direction of its corresponding link. Exactly one of the fundamental loops $\alpha = \tilde{\alpha}$ will include the branch containing $V_s$, and its associated loop current $I_{\tilde{\alpha}} = I_{\rm tot}$, the total current in the network due to $V_s$. The $l = b - N + 1$ equations obtained by writing Kirchhoff's voltage law ($\sum_{q=1}^{n} V_{\alpha_q \alpha_{q+1}} = 0$ for any closed loop) for each fundamental loop associated with the tree $\mathcal{T}$ are linearly independent [16].

(3) Write all branch currents $I_{ij}$ in terms of loop currents $I_\alpha$ only. Each branch current $I_{ij}$ is the sum or difference of a number of loop currents $I_\alpha$.

$$I_{ij} = \sum_\alpha C_{\alpha ij} I_\alpha, \tag{A8}$$

where $C_{\alpha ij} \subset (1,0,-1)$. When the direction of a loop current coincides with the direction of a branch current in the same branch of the network, the coefficient $C_{\alpha ij}$ of the loop current is $+1$. If the loop and branch currents oppose each other in the same branch of the network, the coefficient $C_{\alpha ij}$ of the loop current is $-1$. If the loop and branch currents do not coexist in any branch of the network, the coefficient $C_{\alpha ij}$ of the loop current is zero.

(4) Now consider the term

$$T_3=\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} 2\frac{\partial I_{ij}}{\partial R_{kl}} I_{ij}R_{ij}. \tag{A9}$$

Substitute for $I_{ij}$ in $\partial I_{ij}/\partial R_{kl}$ only, using Eq. (A8),

$$\begin{aligned} T_3&=\sum_{j=1}^{N-1}\sum_{i=j+1}^{N}\sum_{\alpha} 2\frac{\partial(C_{\alpha ij}I_\alpha)}{\partial R_{kl}} I_{ij}R_{ij} \\ &=\sum_{j=1}^{N-1}\sum_{i=j+1}^{N}\sum_{\alpha} 2\frac{\partial I_\alpha}{\partial R_{kl}} C_{\alpha ij}I_{ij}R_{ij}. \end{aligned} \tag{A10}$$

Group all terms which have a given $\partial I_\alpha/\partial R_{kl}$.

$$T_3=\sum_{\alpha} 2\frac{\partial I_\alpha}{\partial R_{kl}}\left(\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} C_{\alpha ij}I_{ij}R_{ij}\right). \tag{A11}$$

Factoring out $\partial I_\alpha/\partial R_{kl}$, we are left with the sum of terms in $C_{\alpha ij}I_{ij}R_{ij}$. Traversing a given loop $\alpha$ in its specified direction while summing voltage drops, we obtain

$$\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} C_{\alpha ij}I_{ij}R_{ij}=0 \tag{A12}$$

for each $\alpha\neq\tilde{\alpha}$ by Kirchhoff's voltage loop rule.

(5) The loop $\alpha=\tilde{\alpha}$ includes the branch containing the source of emf, $V_s$, and has a corresponding loop current $I_{\tilde{\alpha}}=I_{\mathrm{tot}}$. Summing voltage drops around the $\tilde{\alpha}$ loop, one obtains

$$V_s-\sum_{q=1}^{n} I_{\tilde{\alpha}_q\tilde{\alpha}_{q+1}}R_{\tilde{\alpha}_q\tilde{\alpha}_{q+1}}=0. \tag{A13}$$

Since $V_s=I_{\mathrm{tot}}R_{\mathrm{tot}}$ by definition, then

$$I_{\mathrm{tot}}R_{\mathrm{tot}}=\sum_{q=1}^{n} I_{\tilde{\alpha}_q\tilde{\alpha}_{q+1}}R_{\tilde{\alpha}_q\tilde{\alpha}_{q+1}}. \tag{A14}$$

Thus the third term $T_3$ in Eq. (A4) is

$$\begin{aligned} T_3&=\sum_{j=1}^{N-1}\sum_{i=j+1} 2\frac{\partial I_{ij}}{\partial R_{kl}} I_{ij}R_{ij}=2\frac{\partial I_{\tilde{\alpha}}}{\partial R_{kl}}\left(\sum_{j=1}^{N-1}\sum_{i=j+1}^{N} C_{\tilde{\alpha}ij}I_{ij}R_{ij}\right) \\ &=2\frac{\partial I_{\mathrm{tot}}}{\partial R_{kl}} I_{\mathrm{tot}}R_{\mathrm{tot}} \end{aligned} \tag{A15}$$

and is equivalent to the first term

$$T_1=2\frac{\partial I_{\mathrm{tot}}}{\partial R_{kl}} I_{\mathrm{tot}}R_{\mathrm{tot}} \tag{A16}$$

in Eq. (A4).

Therefore the second and fourth terms in Eq. (A4) are equivalent, and we have

$$\frac{I_{kl}^2}{I_{\mathrm{tot}}^2}=\frac{\partial R_{\mathrm{tot}}}{\partial R_{kl}}. \tag{A17}$$

With regard to the above proof, it does not matter if the assigned branch currents have the same directions as the actual branch currents flowing in the network, since changing the direction of a branch current changes the sign of its $C_{\alpha i}$ coefficient, but when summing the voltage drops around a closed current loop the new sign is taken into account, and the sum is again zero.

## APPENDIX B: NET CHARGE ON A CONDUCTING SPHERE

To obtain the net charge on a conducting sphere which is not connected to a battery and resides in a poorly conducting medium, we begin with the steady current condition, Eq. (2):

$$\int_S \vec{J}(x,y)\cdot d\vec{a}=0 \tag{B1}$$

and the condition $J_{1n}=J_{2n}$ for the normal components of the current density on the interface $S$ between media, where the subscript 1 refers to the poorly conducting fluid and subscript 2 refers to the medium of the conducting sphere.

We consider Ohmic media, for which the current density is

$$\vec{J}_i(x,y)=\sigma_i\vec{E}_i(x,y) \quad (i=1,2), \tag{B2}$$

where $\sigma_i$ is the local isotropic conductivity in medium $i$. Together with the boundary condition $J_{1n}=J_{2n}$, we obtain

$$\sigma_1 E_{1n}(x,y)=\sigma_2 E_{2n}(x,y)=J_n(x,y). \tag{B3}$$

Applying Gauss's law

$$\int_S \vec{D}\cdot d\vec{a}=Q \quad \text{inside } S \tag{B4}$$

for a dielectric medium, we obtain

$$D_{1n}(x,y)-D_{2n}(x,y)=\Lambda_f(x,y), \tag{B5}$$

where $\Lambda_f(x,y)$ is the free charge density on the interface of media 1 and 2. Or, since $\vec{D}_i(x,y)=\epsilon_i\vec{E}_i(x,y)(i=1,2)$,

$$\epsilon_1 E_{1n}(x,y)-\epsilon_2 E_{2n}(x,y)=\Lambda_f(x,y), \tag{B6}$$

and from Eqs. (4) and (B3), we have

$$\left(\frac{\epsilon_1}{\sigma_1}-\frac{\epsilon_2}{\sigma_2}\right)J_n(x,y)=\Lambda_f(x,y). \tag{B7}$$

The total free charge on a conducting sphere,

$$Q_f=\int \Lambda_f(x,y)da=\lim_{\sigma_2\to\infty}\left(\frac{\epsilon_1}{\sigma_1}-\frac{\epsilon_2}{\sigma_2}\right)\int J_n(x,y)da \tag{B8}$$

equals zero by Eq. (B1) for any conducting sphere which is not directly injected with charge (i.e., not connected to an external current source such as a battery).

In addition, the net current $I_{i,\mathrm{net}}$ through any sphere $i$ is zero, and is the sum of all currents between sphere $i$ and the other spheres.

$$I_{i,\mathrm{net}}=\sum_{j,j\neq i} I_{ij}=\int_{S_i} J_n(x,y)\,da=\frac{\sigma_1}{\epsilon_1}Q_f=\frac{\sigma_1}{\epsilon_1}\sum_{j,j\neq i} Q_{ij}\,, \tag{B9}$$

where

$$Q_{ij}=\frac{\epsilon_1}{\sigma_1}I_{ij} \tag{B10}$$

and $Q_f=0$ is the total free charge on a conducting sphere. Since each $Q_{ij}$ is directly proportional to $I_{ij}$, and since $I_{ij}=-I_{ji}$, we find

$$Q_{ij}=-Q_{ji}\,, \tag{B11}$$

that is, the charge induced on the surface of sphere $i$ due to the interaction with sphere $j$ is equal and opposite to the charge induced on the surface of sphere $j$ due to the same interaction.

## APPENDIX C: FORCE-RESISTANCE RELATION FOR TWO PARALLEL INFINITE CYLINDERS

For two circular conductors in two dimensions, the electric field distribution can be found exactly by using the method of images. We consider the analogous problem in three-dimensional cylindrical coordinates and neglect the $\hat{z}$ component, since the field is uniform in this direction. The medium between the cylinders is assumed to be linear and isotropic with permittivity $\epsilon$ and conductivity $\sigma$. The potential outside of two infinite, cylindrical conductors $i$ and $j$ is that of two parallel infinite line charges appropriately placed within the conductor boundaries [13]. In cylindrical coordinates for cylinders with identical radii, the potential is

$$\Phi(\vec{\rho})=\frac{\lambda}{2\pi\epsilon}\log_e\left(\frac{\rho_i}{\rho_j}\right), \tag{C1}$$

where $\lambda$ $(-\lambda)$ is the charge per unit length of each of the image line charges, $\rho_i$ and $\rho_j$ are the respective perpendicular distances from the point $\vec{\rho}$ to each image line charge, and the origin is taken as the midpoint between the image line charges. The centers of the circular conductors are also equidistant from the origin, and the distance between the centers is $l$.

Using Eq. (C1), the electric field at the surface of the conductor at the higher potential is found to be

$$\vec{E}_j=\frac{\lambda a}{2\pi\epsilon R[l/2+R\cos(\varphi)]}\hat{n}_j\,, \tag{C2}$$

where $R$ is the radius of each conductor, $\varphi$ is an angle between a radial vector $\vec{R}_j=R\hat{n}_j$ from the center of the conductor and the outward axis formed by the origin and the center of the same conductor, and

$$a=\sqrt{\left(\frac{l}{2}\right)^2-R^2}=\left(\frac{\left(\frac{\rho_i}{\rho_j}\right)^2-1}{\left(\frac{\rho_i}{\rho_j}\right)^2+1}\right)\frac{l}{2} \tag{C3}$$

is the distance of each image line charge from the origin. The electrostatic force on a conductor may be found from the electric field at its surface [14],

$$\vec{F}_j=\frac{1}{2\epsilon}\int_{S_j} E_j^2\,da_j\hat{n}_j\,, \tag{C4}$$

where $\vec{E}_j$ is the electric field at the surface and $\hat{n}_j$ is its direction. Integrating over the entire surface of the conductor, the force per unit length on a cylindrical conductor is

$$\frac{\vec{F}_j}{L}=-\frac{\lambda^2}{4\pi\epsilon a}\hat{l}, \tag{C5}$$

where $-\hat{l}$ indicates that the force is attractive.

We next consider the resistance between two cylinders. For the example above of two conducting cylinders of equal radii, the resistance of a unit length of the cylinders is [13]

$$R_{ij}=\frac{1}{\pi\sigma}\cosh^{-1}\left(\frac{l_{ij}}{2R}\right), \tag{C6}$$

where $\sigma$ is the conductivity of the medium between the conductors and

$$l_{ij}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2} \tag{C7}$$

is the distance between the centers of the conductors in terms of two-dimensional Cartesian coordinates.

We now consider the gradient of the resistance $R_{ij}$ with respect to conductor location $\vec{r}_i=x_i\hat{x}+y_i\hat{y}$:

$$\vec{\nabla}_{r_i}R_{ij}=\frac{\partial}{\partial r_i}\left[\frac{1}{\pi\sigma}\cosh^{-1}\left(\frac{l_{ij}}{2R}\right)\right]\hat{r}_i=\frac{1}{2\pi\sigma}\,\frac{1}{\sqrt{\left(\frac{l_{ij}}{2}\right)^2-R^2}}\hat{l}_{ij}$$

$$=\frac{1}{2\pi\sigma a}\hat{l}_{ij}\,, \tag{C8}$$

where we have used Eq. (C3) and $\hat{l}_{ij}$ is a unit vector along the direction of the displacement vector between the centers of the conductors:

$$\hat{l}_{ij}=\frac{(x_i-x_j)\hat{x}+(y_i-y_j)\hat{y}}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}. \tag{C9}$$

From Eqs. (C5), (C8), and $\lambda=Q/L$, the force on conductor $i$ at location $\vec{x}_i$ due to the field $\vec{E}_{ij}$ may be written

$$\vec{F}_{ij}=-\frac{\sigma}{2\epsilon}Q_{ij}^2\vec{\nabla}_{r_i}R_{ij}\,. \tag{C10}$$

And from Eq. (B10) we have

$$\vec{F}_{ij}=-\frac{\epsilon}{2\sigma}I_{ij}^2\vec{\nabla}_{r_i}R_{ij} \tag{C11}$$

as the force on conductor $i$ due to its interaction with conductor $j$.

## APPENDIX D: FORCE-RESISTANCE RELATION FOR A POINT CHARGE AND A GROUNDED, CONDUCTING SPHERE

For the problem of an isolated point charge and a grounded conducting sphere, the force on the point charge may be determined by considering the gradient of a resistancelike quantity, $\mathcal{R}_{\rm tot}$. In order to demonstrate this method, we first solve for the electric field everywhere outside the sphere by using the appropriate image charge. We consider a point charge $q_0>0$ located at $\vec{x}_0$ and a sphere of radius $r_s$ centered at $\vec{x}_s$ where $|\vec{x}_0|>|\vec{x}_s|$. The electric field outside the sphere is determined in the standard way by placing an image charge $q'=-r_s q_0/(x_0-x_s)$ a distance $\eta=r_s^2/(x_0-x_s)$ from $\vec{x}_s$ in the $\hat{x}$ direction. The location of $q_0$ and $q'$ satisfies the boundary conditions that $\Phi(\vec{r})=0$ on the surface of the sphere and at infinity. The force between the point charge and sphere may now be found directly using Coulomb's law:

$$\vec{F}=\frac{1}{4\pi\epsilon_0}\frac{q_0 q'}{(x_0-x_s-\eta)^2}\hat{x}=-\frac{1}{4\pi\epsilon_0}\frac{r_s q_0^2(x_0-x_s)}{[(x_0-x_s)^2-r_s^2]^2}\hat{x}. \tag{D1}$$

Alternatively, we may determine the force between the point charge and sphere through the introduction of the resistancelike scalar quantity

$$\mathcal{R}_{\rm tot}=\frac{\int_a^b \vec{E}\cdot d\vec{r}}{\int_{S_{\rm tot}}\vec{E}\cdot d\vec{a}}, \tag{D2}$$

where $\vec{E}=-\vec{\nabla}\Phi(\vec{r})$ is the electrostatic field outside the sphere, $S_{\rm tot}$ is a closed surface about $q_0$, and $a$ and $b$ are end points of the total field configuration [i.e., the location of the point charge, $\vec{x}_0$, and any point on a boundary where $\Phi(\vec{r})=0$]. From $\vec{x}_0$, the total vector field (the vector sum of the electrostatic fields due to $q_0$ and $q'$) approaches one of two possible boundaries: the surface of the conducting sphere or the boundary at infinity. The total vector field may be divided into two subspaces corresponding to the regions of the total field which terminate at one boundary or the other. We then determine the quantity $\mathcal{R}$ for each of these subspaces. In keeping with the idea that $\mathcal{R}$ is resistancelike, the integral $\int\vec{E}\cdot d\vec{r}$ must be the same for each subspace. The $\int_S\vec{E}\cdot d\vec{a}$ terms may be determined in this case from Gauss's law. $d\vec{r}$ and $d\vec{a}$ are defined along the direction of the total electric field $\vec{E}$, such that each integrand is non-negative.

For the subspace of the total vector field which terminates on the surface of the conducting sphere (i.e., the region of the electrostatic field between $q_0$ and $q'$), we define the quantity

$$\mathcal{R}_0'=\frac{\int_a^b\vec{E}\cdot d\vec{r}}{\int_{S_0'}\vec{E}\cdot d\vec{a}}=\frac{\epsilon_0}{(-q')}\lim_{\delta\to 0}\int_{x_s+r_s}^{x_0-\delta}\frac{1}{4\pi\epsilon_0}\left[\frac{q_0\,dx}{(x_0-x)^2}+\frac{-q'\,dx}{(x-x_s-\eta)^2}\right]=\frac{1}{4\pi}\frac{(x_0-x_s)}{r_s q_0}\lim_{\delta\to 0}\left[\frac{q_0}{\delta}-\frac{q_0}{(x_0-x_s-r_s)}\right.$$
$$\left.-\frac{r_s q_0}{(x_0-x_s)(x_0-x_s-\eta)}+\frac{r_s q_0}{(x_0-x_s)(r_s-\eta)}\right]=\frac{1}{4\pi}\lim_{\delta\to 0}\left[\frac{x_0-x_s}{r_s\delta}-\frac{(x_0-x_s)}{(x_0-x_s)^2-r_s^2}\right], \tag{D3}$$

where we have used $\eta=r_s^2/(x_0-x_s)$. We have taken $\vec{x}_0$ as the limit $a$ of the integral $\int_a^b\vec{E}\cdot d\vec{r}=-\int_a^b\vec{E}\cdot d\vec{x}$ by introducing the parameter $\delta$ and taking the limit $\delta\to 0$. This definition introduces a singularity in $\mathcal{R}_0'$, but does not lead to a divergence in the expression for the force. The other limit of this integral, $b=x_s+r_s$, corresponds to the point where the $x$ axis intersects the surface of the sphere, where $\Phi(x_s\pm r_s)=0$. The integral $\int_{S_0'}\vec{E}\cdot d\vec{a}$, taken over the region of the vector space which terminates on the surface of the sphere, is equal to $-q'/\epsilon_0$ from Gauss's law, by taking a Gaussian surface around the conducting sphere.

For the subspace of the total vector field which extends to infinity, we define the quantity

$$\mathcal{R}_0^\infty=\frac{\int_a^c\vec{E}\cdot d\vec{r}}{\int_{S_0^\infty}\vec{E}\cdot d\vec{a}}=\frac{\epsilon_0}{(q_0+q')}\lim_{\delta\to 0}\int_{x_0+\delta}^{\infty}\frac{1}{4\pi\epsilon_0}\left[\frac{q_0\,dx}{(x_0-x)^2}+\frac{q'\,dx}{(x-x_s-\eta)^2}\right]$$
$$=\frac{1}{4\pi}\frac{(x_0-x_s)}{q_0(x_0-x_s-r_s)}\lim_{\delta\to 0}\left[\frac{q_0}{\delta}-\frac{r_s q_0}{(x_0-x_s)(x_0-x_s-\eta)}\right]=\frac{1}{4\pi}\lim_{\delta\to 0}\left[\frac{x_0-x_s}{(x_0-x_s-r_s)\delta}-\frac{r_s(x_0-x_s)}{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)}\right], \tag{D4}$$

where we have again used $\eta=r_s^2/(x_0-x_s)$. The limits of integration are $a=\lim_{\delta\to 0}(x_0+\delta)$ and $c=\infty$. The integral $\int_{S_0^\infty}\vec{E}\cdot d\vec{a}$, corresponding to the subregion of the total vector field which extends to infinity, is equal to $(q_0+q')/\epsilon_0$ by taking a Gaussian surface around both charges.

Next we show that $\mathcal{R}_{\rm tot}$ has resistancelike properties. If we consider $\mathcal{R}_0'$ and $\mathcal{R}_0^\infty$ as quantifying the flux of electrostatic field in different subspaces of the total field but between the same potential differences, we may think of these quantities as resistors in parallel. Then the total ''field resistance'' of the system may be defined by

$$\frac{1}{\mathcal{R}_{\rm tot}}=\frac{1}{\mathcal{R}_0'}+\frac{1}{\mathcal{R}_0^{\infty}}=\frac{\int_{S_0'}\vec{E}\cdot d\vec{a}}{\int_a^b\vec{E}\cdot d\vec{r}}+\frac{\int_{S_0^{\infty}}\vec{E}\cdot d\vec{a}}{\int_a^c\vec{E}\cdot d\vec{r}}=\frac{\int_{S_{\rm tot}}\vec{E}\cdot d\vec{a}}{\int_a^b\vec{E}\cdot d\vec{r}},\tag{D5}$$

where we have used

$$\int_a^b\vec{E}\cdot d\vec{r}=-\int_a^b\vec{E}\cdot d\vec{x}=\int_a^c\vec{E}\cdot d\vec{x}=\int_a^c\vec{E}\cdot d\vec{r}\tag{D6}$$

and

$$\int_{S_0'}\vec{E}\cdot d\vec{a}+\int_{S_0^{\infty}}\vec{E}\cdot d\vec{a}=-\frac{q'}{\epsilon_0}+\frac{(q_0+q')}{\epsilon_0}=\frac{q_0}{\epsilon_0}=\int_{S_{\rm tot}}\vec{E}\cdot d\vec{a}.\tag{D7}$$

The force between the point charge and the sphere may be expressed in terms of the gradient of $\mathcal{R}_{\rm tot}$ with respect to the point charge location $\vec{x}_0$ or the sphere location $\vec{x}_s$. The gradient of $\mathcal{R}_{\rm tot}$ with respect to $\vec{x}_0$ is

$$\vec{\nabla}_{x_0}\mathcal{R}_{\rm tot}=\left(\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0'}\frac{\partial\mathcal{R}_0'}{\partial x_0}+\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0^{\infty}}\frac{\partial\mathcal{R}_0^{\infty}}{\partial x_0}\right)\hat{x}.\tag{D8}$$

We now proceed to evaluate the partial derivatives in the above expression.

From Eq. (D5), $\mathcal{R}_{\rm tot}$ is given by

$$\mathcal{R}_{\rm tot}=\frac{\mathcal{R}_0'\mathcal{R}_0^{\infty}}{\mathcal{R}_0'+\mathcal{R}_0^{\infty}}.\tag{D9}$$

The partial derivatives of $\mathcal{R}_{\rm tot}$ with respect to $\mathcal{R}_0'$ and $\mathcal{R}_0^{\infty}$ are

$$\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0'}=\frac{(\mathcal{R}_0^{\infty})^2}{(\mathcal{R}_0'+\mathcal{R}_0^{\infty})^2}=\frac{\left(\frac{1}{4\pi}\lim_{\delta\to0}\left[\frac{(x_0-x_s)}{(x_0-x_s-r_s)\delta}-\frac{r_s(x_0-x_s)}{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)}\right]\right)^2}{\left(\frac{1}{4\pi}\lim_{\delta\to0}\left[\frac{(x_0-x_s)^2}{r_s(x_0-x_s-r_s)\delta}-\frac{(x_0-x_s)^2}{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)}\right]\right)^2}=\frac{r_s^2}{(x_0-x_s)^2}=\left(\frac{q'}{q_0}\right)^2\tag{D10}$$

and

$$\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0^{\infty}}=\frac{(\mathcal{R}_0')^2}{(\mathcal{R}_0'+\mathcal{R}_0^{\infty})^2}=\frac{\left(\frac{1}{4\pi}\lim_{\delta\to0}\left[\frac{(x_0-x_s)}{r_s\delta}-\frac{(x_0-x_s)}{[(x_0-x_s)^2-r_s^2]}\right]\right)^2}{\left(\frac{1}{4\pi}\lim_{\delta\to0}\left[\frac{(x_0-x_s)^2}{r_s(x_0-x_s-r_s)\delta}-\frac{(x_0-x_s)^2}{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)}\right]\right)^2}=\frac{(x_0-x_s-r_s)^2}{(x_0-x_s)^2}=\left(\frac{q'+q_0}{q_0}\right)^2.\tag{D11}$$

The partial derivatives of $\mathcal{R}_0'$ and $\mathcal{R}_0^{\infty}$ with respect to $x_0$ are

$$\frac{\partial\mathcal{R}_0'}{\partial x_0}=\frac{1}{4\pi}\lim_{\delta\to0}\left[\frac{1}{r_s\delta}+\frac{(x_0-x_s)^2+r_s^2}{[(x_0-x_s)^2-r_s^2]^2}\right]\tag{D12}$$

and

$$\frac{\partial\mathcal{R}_0^{\infty}}{\partial x_0}=\frac{1}{4\pi}\lim_{\delta\to0}\left[-\frac{r_s}{\delta(x_0-x_s-r_s)^2}+\frac{2r_s(x_0-x_s)^3-r_s^2(x_0-x_s)^2-r_s^4}{\{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)\}^2}\right].\tag{D13}$$

We now consider the product of Eq. (D8) with the square of the magnitude of the point charge:

$$q_0^2\vec{\nabla}_{x_0}\mathcal{R}_{\rm tot}=q_0^2\left(\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0'}\frac{\partial\mathcal{R}_0'}{\partial x_0}+\frac{\partial\mathcal{R}_{\rm tot}}{\partial\mathcal{R}_0^\infty}\frac{\partial\mathcal{R}_0^\infty}{\partial x_0}\right)\hat{x}=q_0^2\left(\frac{(q')^2}{q_0^2}\frac{1}{4\pi}\lim_{\delta\to 0}\left[\frac{1}{r_s\delta}+\frac{(x_0-x_s)^2+r_s^2}{[(x_0-x_s)^2-r_s^2]^2}\right]+\frac{(q'+q_0)^2}{q_0^2}\frac{1}{4\pi}\lim_{\delta\to 0}\right.$$
$$\left.\times\left[-\frac{r_s}{\delta(x_0-x_s-r_s)^2}+\frac{2r_s(x_0-x_s)^3-r_s^2(x_0-x_s)^2-r_s^4}{\{[(x_0-x_s)^2-r_s^2](x_0-x_s-r_s)\}^2}\right]\right)\hat{x}=\frac{1}{4\pi}\left[\frac{2r_s(x_0-x_s)q_0^2}{[(x_0-x_s)^2-r_s^2]^2}\right]\hat{x}. \tag{D14}$$

Comparing the above equation for $q_0^2\vec{\nabla}_{x_0}\mathcal{R}_{\rm tot}$ with Eq. (D1), we find that the force on the point charge may be written

$$\vec{F}_{q_0}=-\frac{q_0^2}{2\epsilon_0}\vec{\nabla}_{x_0}\mathcal{R}_{\rm tot}. \tag{D15}$$

We now consider the case where the point charge $q_0$ and grounded conducting sphere exist in a linear, isotropic, homogeneous medium of permittivity $\epsilon$ and conductivity $\sigma$. The point charge $q_0$ is now considered as a current source, e.g., a point electrode in the experimental system. If the time scale of the motion of the sphere or electrode is large compared to the time scale of the relaxation of the charge distribution $\epsilon/\sigma$, we may neglect the time dependence of the charge density $\rho(x,y)$ in the medium. The steady current condition, Eq. (1), along with Ohm's law, $\vec{J}_i(x,y)=\sigma_i\vec{E}_i(x,y)$, implies the surface charge density-current relation, Eq. (B10),

$$q_{\rm sphere}=\frac{\epsilon}{\sigma}I_{\rm sphere}. \tag{D16}$$

The actual resistance between the conducting sphere and point electrode is

$$R_{\rm tot}=\frac{\Delta\Phi_{\rm tot}}{I_{\rm tot}}=\frac{\int_p^{\rm SS}\vec{E}\cdot d\vec{r}}{\int_S\vec{J}\cdot d\vec{a}}=\frac{\int_p^{\rm SS}\vec{E}\cdot d\vec{r}}{\sigma\int_S\vec{E}\cdot d\vec{a}} \tag{D17}$$

and is related to the field resistance $\mathcal{R}_{\rm tot}$ by

$$R_{\rm tot}=\frac{\mathcal{R}_{\rm tot}}{\sigma}, \tag{D18}$$

where $\Delta\Phi_{\rm tot}$ is the total potential difference between the point electrode and the sphere, $I_{\rm tot}$ is the total current between them, $p$ is the location of the point electrode, SS is a point on the surface of the sphere, and $S$ is the entire surface of the sphere. In terms of the physical resistance $R_{\rm tot}$, the force between a conducting sphere and a point electrode is estimated as

$$\vec{F}_{q_{\rm tot}}=-\frac{q_{\rm tot}^2}{2\epsilon}\vec{\nabla}_{x_0}\mathcal{R}_{\rm tot}=-\frac{\epsilon}{2\sigma}I_{\rm tot}^2\vec{\nabla}_{x_0}R_{\rm tot}. \tag{D19}$$


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